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Thermal field around a partially debonded spherical particle

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Abstract

The thermal field around a partially debonded spherical particle, having a spherical-cap crack, embedded in an infinite matrix is presented. Heat flow parallel and perpendicular to the axis of symmetry is considered. When the conductivity of the particle is equal to that of the matrix the results of the present analysis agree with previously published results. An estimate for the effective thermal conductivity of a dispersion of these particles is presented.

Keywords: Mixed boundary conditions; Heat conduction in composites

1. Introduction

There has been a significant amount of research directed toward finding the effective thermal conductivity of composites; a more complete discussion is given in the two recently published books by Torquato [1], and by Sahimi [2]. Most of the attention has been directed toward two phase composite solids, comprising particles with thermal conductivity k_p embedded in a matrix with conductivity k_m. Early estimates for the effective conductivity of a composite containing spherical particles, where each particle is in perfect contact with the matrix, has been given for a random dispersion by Maxwell [3], and, for a periodic arrangement of spherical particles, by Raleigh [4]. Subsequently, both the Maxwell and Raleigh models have been modified, see Refs. [5,6], respectively, to account for the presence of a thin thermal barrier that completely surrounds the particle.

In this note the effective thermal conductivity of a random dispersion of partially debonded particles is considered. The flow field around a single particle embedded in an infinite matrix is the starting point used in most models that are employed to find the effective conductivity. A spherical particle is used and its interface with the matrix has mixed boundary conditions; there is a spherical-cap crack covering the top portion of the sphere while the remainder of the interface is in perfect contact. This problem has received far less attention than the calculation where the interface condition is constant everywhere on the interface. When the thermal conductivity of particle is equal to that of the matrix, the calculation reduces to the Neumann spherical-cap problem; the solution has been given by Collins [7] and more recently by Martin [8]. The temperature around a particle having a conductivity different from that of the matrix is not readily available in the literature. This thermal field is presented here and is then used to estimate the effective conductivity corresponding to a finite volume fraction of these partially debonded particles.

The paper proceeds as follows: (a) in Section 2 the mathematical statement of the problem is presented; (b) in Section 3 the thermal field perpendicular to the axis of symmetry is derived; (c) Section 4 deals with the axisymmetric flow perpendicular to the cap; and (d) in Section 5 an estimate for the effective conductivity is given.

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Nomenclature

a	radius of the particle
A_n, B_n	Fourier coefficients in Eq. (6)
$C_n(x), S_n(x)$	functions in Eq. (18)
$f(x)$	function in Section 3
F_n, G_n, I_{nm}	integrals in Section 3
$\mathscr{F}_n, \mathscr{G}_n$	coefficients in Section 3
H_n	coefficient given in Eq. (7)
k_p, k_m	thermal conductivity of the particle and matrix
k_x^e, k_z^e	effective anisotropic conductivity with aligned particles
$P_n(\cos\theta)$	Legendre polynomials
$P_n^1(\cos\theta)$	associated Legendre functions
q_a	radial flux across the interface
u^∞	far-field temperature in the matrix
u, v	temperatures in the matrix and particle
x, y, z	Cartesian coordinates

Greek symbols

α	semi-angle of the debonded cap
$\beta = k_m/k_p$	conductivity ratio
γ	polarizability given in Eq. (57)
δ_a	temperature jump across the interface
ε_x	far-field temperature gradient in x direction
ε_z	far-field temperature gradient in z direction
η	integration constant in Section 3
$\mu = \cos\theta$	abbreviation
ξ	constant appearing in Eq. (11)
ρ, θ, ϕ	spherical coordinates
ω	constant appearing in Eq. (15)

2. Partially debonded particle

To obtain an expression for the effective conductivity, the thermal field around a partially debonded particle embedded in an infinite matrix needs to be found. It is therefore expedient to use spherical coordinates (r, θ, ϕ). Consider an embedded, partially debonded, spherical particle with radius a where the portion $0 < \theta < \alpha$ is debonded and there is no heat flow across this debonded cap. The geometry is axisymmetric with respect to the z axis. The particle has thermal conductivity k_p while the matrix has thermal conductivity k_m and it is is convenient to define the conductivity ratio k_m/k_p as

$$\beta = \frac{k_m}{k_p}. \tag{1}$$

The temperatures within the matrix and particle are u, v respectively, and at steady state these temperatures must satisfy Laplace's equation, $\nabla^2 u = \nabla^2 v = 0$. The boundary conditions at $r = a$ are written as

$$k_m \frac{\partial u}{\partial r} = k_p \frac{\partial v}{\partial r}, \quad 0 < \theta < \pi, \tag{2}$$

$$u = v, \quad \alpha < \theta < \pi, \tag{3}$$

$$k_m \frac{\partial u}{\partial r} = k_p \frac{\partial v}{\partial r} = 0, \quad 0 < \theta < \alpha. \tag{4}$$

Over the perfect portion the temperatures are matched, while the debonded region is adiabatic.

At very large distances from the particle, i.e. when $r \gg a$, the matrix temperature field corresponds to uniform applied heat flow where the temperature u^∞ is written as the linear form

$$u^\infty = \varepsilon_x x + \varepsilon_z z. \tag{5}$$

Because of the symmetry only the flow components in the x and z directions need to be considered. Near the particle the far-field is disturbed and temperature in the matrix is written as

$$u = u^\infty - a\sum_{n=1}^{\infty}(1 + H_n)n\left(\frac{a}{r}\right)^{1+n} \times \left\{\varepsilon_x \cos\phi A_n P_n^1(\mu) + \varepsilon_z B_n P_n(\mu)\right\}, \tag{6}$$

where $\mu = \cos\theta$, $P_n(\mu)$ is Legendre's polynomial, and

$$P_n^1(\mu) = \sqrt{1-\mu^2}\frac{\mathrm{d}P_n(\mu)}{\mathrm{d}\mu}$$

is the associated Legendre's function. The coefficient H_n is given by

$$H_n = \frac{1-\beta}{2(n + n\beta + \beta)}. \tag{7}$$

In the expression for the matrix temperature there is no series term corresponding to $n = 0$ because there is no internal generation of heat. The average temperature within the particle is not zero and the expression for v is written as

$$v = \beta\left\{u^\infty + a\varepsilon_z(1 + H_0)B_0 + a\sum_{n=1}^{\infty}(1 + H_n)(1 + n)\left(\frac{r}{a}\right)^n \times \left[\varepsilon_x \cos\phi A_n P_n^1(\mu) + \varepsilon_z B_n P_n(\mu)\right]\right\}. \tag{8}$$

It is noted that the temperatures u and v have been especially written so that there is continuity of radial flux across the particle-matrix interface and hence Eq. (2) is identically satisfied.

At $r = a$ it is possible to find expressions for the radial flux and the temperature jump across the interface. The radial flux across the interface at $r = a$ is written as

$$-q_a = k_m\varepsilon_x\left\{\sum_{n=1}^{\infty} n(1+n)(1+H_n)A_n P_n^1(\cos\theta) + P_1^1(\cos\theta)\right\}\cos\phi + k_m\varepsilon_z\left\{\sum_{n=1}^{\infty} n(1+n)(1+H_n)B_n P_n(\cos\theta) + P_1(\cos\theta)\right\}. \tag{9}$$

The temperature jump across the interface, $\delta_a \equiv u(a) - v(a)$, is written as

$$\delta_a = -\left(\frac{1+\beta}{2}\right)a\varepsilon_x\left\{\xi P_1^1(\cos\theta) + \sum_{n=1}^{\infty}(1+2n)A_nP_n^1(\cos\theta)\right\}\cos\phi - \left(\frac{1+\beta}{2}\right)a\varepsilon_z\left\{\xi P_1(\cos\theta) + \sum_{n=0}^{\infty}(1+2n)B_nP_n(\cos\theta)\right\}, \tag{10}$$

where the constant

$$\xi = \frac{2(\beta-1)}{1+\beta}. \tag{11}$$

The average particle temperature depends on the direction of the axial heat flow and the particle either expands or contracts depending on its average temperature. Thus, the direction of heat flow might have possible implications with regard to tensile stress at the interface.

In summary, over the cap-crack, $\alpha > \theta > 0$, there is no radial flux so that $q_a = 0$, while over the perfect portion of the interface, $\pi > \theta > \alpha$, there is no temperature jump so that $\delta_a = 0$. These are the so-called mixed boundary conditions and their solution has been considered in the seminal work by Collins [7]. However, the treatment used by Collins is difficult to follow and here, although we use many of his suggestions, a different approach is employed to solve the dual series. In the special case when $\beta = 1$ the expressions for A_n and B_n given in Eqs. (35) and (49) agree with his results.

3. Transverse flow

The coefficient A_n is associated with the heat flow in the x direction. The boundary condition describing the radial flux, which is found using Eq. (9), is given by

$$\sum_{n=1}^{\infty} n(1+n)(1+H_n)A_nP_n^1(\cos\theta) + P_1^1(\cos\theta) = 0, \quad 0 < \theta < \alpha. \tag{12}$$

Now it is known that the Legendre functions satisfy the differential equation

$$n(1+n)P_n^1(\cos\theta) = \mathscr{D}_1P_n^1(\cos\theta), \tag{13}$$

where the operator is

$$\mathscr{D}_1 = \frac{1}{\sin^2\theta} - \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right).$$

Applying this differential equation the series written in Eq. (12) can be replaced by

$$\sum_{n=1}^{\infty}(1+H_n)A_nP_n^1(\cos\theta) + \frac{P_1^1(\cos\theta)}{2} = -2\omega\eta\tan\left(\frac{\theta}{2}\right), \quad 0 < \theta < \alpha, \tag{14}$$

where $\mathscr{D}_1\tan(\theta/2) = \mathscr{D}_1\cot(\theta/2) = 0$ and the complementary solution $\cot(\theta/2)$ is rejected since it is infinite when $\theta = 0$. The constant η is unknown at this stage and the term

$$\omega = \frac{-3}{2(1+2\beta)}. \tag{15}$$

Over the remaining portion of the interface there is perfect contact and the boundary condition in Eq. (10) is

$$\sum_{n=1}^{\infty}(1+2n)A_nP_n^1(\cos\theta) + \xi P_1^1(\cos\theta) = 0, \quad \alpha < \theta < \pi. \tag{16}$$

The boundary conditions are given by the dual series in Eqs. (14) and (16) so the problem then becomes to find the coefficients $A_1, A_2, A_3, \ldots$ that satisfy these two series.

3.1. Some mathematical results

In the subsequent analysis, integral representation for the associated Legendre functions are required. Similar to the Mehler–Dirichlet representations for $P_n(\cos\theta)$ the representations for Legendre's function are written as

$$P_n^1(\cos\theta) = -\frac{\sqrt{2}}{\pi}\cot\left(\frac{\theta}{2}\right)\int_0^{\theta}\frac{C_n(x)\mathrm{d}x}{\sqrt{\cos x - \cos\theta}},$$
$$P_n^1(\cos\theta) = -\frac{\sqrt{2}}{\pi}\tan\left(\frac{\theta}{2}\right)\int_{\theta}^{\pi}\frac{S_n(x)\mathrm{d}x}{\sqrt{\cos\theta - \cos x}}, \tag{17}$$

where the functions $C_n(x)$ and $S_n(x)$ are given by

$$C_n(x) = \sin\left(\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos(n+1/2)x}{\cos(x/2)}\right) = -2\sin\left(\frac{x}{2}\right)\sum_{p=1}^{n}(-1)^{n+p}p\sin px,$$
$$S_n(x) = \cos\left(\frac{x}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(n+1/2)x}{\sin(x/2)}\right) = -2\cos\left(\frac{x}{2}\right)\sum_{p=1}^{n}p\sin px. \tag{18}$$

The general explanation for the derivation of these integral expressions is described in Chapter 2 of the book by Sneddon [9].

In addition to these integral representations some further identities are required and the two results in Eqs. (20) and (21) are essential for the treatment used here. The integral

$$I_{nm} \equiv \int_0^{\alpha}\cot^2\left(\frac{t}{2}\right)C_n(t)C_m(t)\mathrm{d}t = \frac{1}{4}\left[2\tan\zeta\cos N\zeta\cos M\zeta - \frac{(1+MN)\sin(N+M)\zeta}{N+M} - \frac{(1-MN)\sin(N-M)\zeta}{N-M}\right], \tag{19}$$

where $\zeta = \alpha/2$, $M = 2m+1$, and $N = 2n+1$. If $n = m$ then l'Hôpital's rule is used and when $\alpha = \pi$ it is clear that the functions $C_n(x)$ possess the following orthogonality relation

$$\int_0^{\pi} \cot^2\left(\frac{t}{2}\right) C_n(t) C_m(t)\mathrm{d}t = \frac{\pi}{2}\begin{cases} n(1+n), & n=m, \\ 0, & n \neq m. \end{cases} \tag{20}$$

Also the following results are required:

$$S_n(x) = -\frac{2}{2n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\cot^2\left(\frac{x}{2}\right)C_n(x)\right\}, \tag{21}$$

$$\begin{aligned} G_n &= \int_0^{\alpha} \cot^2\left(\frac{t}{2}\right) C_n(t) C_1(t)\mathrm{d}t \\ &= \frac{1}{2}\left[\left(\frac{n+1}{n-1}\right)\sin(n-1)\alpha + \sin n\alpha - \sin(n+1)\alpha \right. \\ &\quad \left. -\left(\frac{n}{n+2}\right)\sin(n+2)\alpha\right], \end{aligned} \tag{22}$$

$$\begin{aligned} F_n &= \int_0^{\alpha} \cot^2\left(\frac{x}{2}\right) C_n(x) f(x)\mathrm{d}x \\ &= \sum_{p=1}^{n}(-1)^{n+p}\left[2\alpha\cos p\alpha - \frac{2\sin p\alpha}{p} \right. \\ &\quad \left. -\frac{p\sin(p-1)\alpha}{p-1} + \frac{p\sin(p+1)\alpha}{p+1}\right], \end{aligned} \tag{23}$$

where $f(x)$ is defined in Eq. (26). Integration by parts has been used to obtain these results and l'Hôpital's rule is used when necessary.

3.2. Solution of the dual equations

Using the first integral representation in Eq. (17) the series in Eq. (14) is expressed as

$$\begin{aligned} &\int_0^{\theta} \frac{\mathrm{d}x}{\sqrt{\cos x - \cos\theta}}\left\{\sum_{n=1}^{\infty}(1+H_n)A_n C_n(x) + \frac{C_1(x)}{2}\right\} \\ &= \frac{2\pi\omega\eta}{\sqrt{2}}\tan^2\left(\frac{\theta}{2}\right), \quad 0<\theta<\alpha. \end{aligned} \tag{24}$$

The solution to an integral equation of this type is given on page 41 of the book by Sneddon [9]

$$\sum_{n=1}^{\infty}(1+H_n)A_n C_n(x) + \frac{C_1(x)}{2} = \omega\eta f(x), \tag{25}$$

where

$$f(x) = \frac{\sin(x/2)(x+\sin x)}{\cos^2(x/2)}. \tag{26}$$

Next, the series in Eq. (16) represents the fact that there is no temperature jump across $r = a$ in the region $\alpha < \theta < \pi$. Using the second integral representations in Eq. (17) this series can be written as

$$\begin{aligned} &\int_{\theta}^{\pi} \frac{\mathrm{d}x}{\sqrt{\cos\theta - \cos x}}\left\{\sum_{n=1}^{\infty}(1+2n)A_n S_n(x) + \xi S_1(x)\right\} = 0, \\ &\alpha<\theta<\pi. \end{aligned} \tag{27}$$

In this integral the portion within $\{\cdots\}$ is equal to zero so that

$$\sum_{n=1}^{\infty}(1+2n)A_n S_n(x) + \xi S_1(x) = 0.$$

After applying the result given in Eq. (21), the expression

$$-2\cot^2\left(\frac{x}{2}\right)\left[\sum_{n=1}^{\infty}A_n C_n(x) + \frac{\xi C_1(x)}{3}\right] = C, \quad \alpha<x<\pi$$

is obtained where C is a constant. Since the left hand side of this equation tends to zero as x approaches π it follows that the constant $C = 0$. The transformed dual series are then expressed as

$$\sum_{n=1}^{\infty}(1+H_n)A_n C_n(x) + \frac{C_1(x)}{2} = \omega\eta f(x), \quad 0<x<\alpha, \tag{28}$$

$$\sum_{n=1}^{\infty}A_n C_n(x) + \frac{\xi C_1(x)}{3} = 0, \quad \alpha<x<\pi. \tag{29}$$

To solve these equations a function $h(x)$ is introduced by extending the latter series into the region $0 < x < \alpha$ to obtain

$$\sum_{n=1}^{\infty}A_n C_n(x) + \frac{\xi C_1(x)}{3} = \omega\tan\left(\frac{x}{2}\right)\begin{cases} h(x), & 0<x<\alpha \\ 0, & \alpha<x<\pi. \end{cases} \tag{30}$$

Using the orthogonality condition in Eq. (20), the coefficient A_n is found as

$$A_n = -\frac{\xi\delta_{1n}}{3} + \frac{2\omega}{\pi n(1+n)}\int_0^{\alpha} h(t)\cot\left(\frac{t}{2}\right)C_n(t)\mathrm{d}t. \tag{31}$$

For large values of n the Legendre's function $P_n^1(\cos\theta)$ is $O(\sqrt{n})$ in any closed subinterval in the range $0 < \theta < \pi$, so that if the series in Eq. (12) is to converge, the coefficient A_n must be $O(n^{-3})$. The function $h(x) \to 0$ as $x \to 0$ and to ensure convergence of the series in Eq. (12), the constraint

$$h(\alpha) = 0 \tag{32}$$

is placed upon the function $h(x)$ and eventually η is determined by this condition.

The function $h(x)$ is found by solving the integral equation that emerges by substituting expressions (30) and (31) into (28) to obtain

$$\begin{aligned} &h(x) + \frac{2}{\pi}\int_0^{\alpha} h(t)K(x,t)\mathrm{d}t = \cot\left(\frac{x}{2}\right)\{C_1(x) + \eta f(x)\}, \\ &0<x<\alpha, \end{aligned} \tag{33}$$

and the kernel is

$$K(x,t) = \cot\left(\frac{x}{2}\right)\cot\left(\frac{t}{2}\right)\sum_{n=1}^{\infty}\frac{H_n C_n(x) C_n(t)}{n(1+n)}. \tag{34}$$

When $\beta = 1$ the kernel $K(x,t) = 0$ so that the function $h(x)$ is equal to the expression on the right hand side of Eq. (33). For values of $\beta \approx 1$ this can be regarded as the first term of an iterated solution. If $h(\alpha) = 0$ it is required that the coefficient $\eta = \eta_0 = -C_1(\alpha)/f(\alpha)$ so that $\eta_0 = 2\sin\alpha\cos^2(\alpha/2)/(\alpha + \sin\alpha)$. The coefficients A_n are then given by

$$A_n \approx -\frac{\xi\delta_{1n}}{3} + \frac{2\omega(G_n + \eta_0 F_n)}{\pi n(1+n)}. \tag{35}$$

When $\beta = 1$ this expression for A_n is in agreement with Collins' result.

3.3. Approximate solution

In general it is not possible to find the exact solution for A_n so that some approximate solution is required. Since the integral equation is linear, it is possible to write

$$h(x) = h_0(x) + \eta h_1(x) \tag{36}$$

and then find these two functions separately; Eq. (32) is then used to find η. Unknown quantities $\mathscr{F}_n$ and $\mathscr{G}_n$ are defined by

$$\begin{aligned} \mathscr{G}_n &\equiv \int_0^\alpha h_0(t)\cot\left(\frac{t}{2}\right)C_n(t)\mathrm{d}t \\ \mathscr{F}_n &\equiv \int_0^\alpha h_1(t)\cot\left(\frac{t}{2}\right)C_n(t)\mathrm{d}t \end{aligned} \tag{37}$$

The coefficients A_n are then expressed in terms of these unknown quantities

$$A_n = -\frac{\xi\delta_{1n}}{3} + \frac{2\omega(\mathscr{G}_n + \eta\mathscr{F}_n)}{\pi n(1+n)}. \tag{38}$$

If the kernel is approximated by truncating the series at N terms, then the two unknown functions are approximated as

$$\begin{aligned} h_0(x) + \frac{2}{\pi}\cot\left(\frac{x}{2}\right)\sum_{n=1}^{N}\frac{H_nC_n(x)\mathscr{G}_n}{n(1+n)} &\approx \cot\left(\frac{x}{2}\right)C_1(x), \\ h_1(x) + \frac{2}{\pi}\cot\left(\frac{x}{2}\right)\sum_{n=1}^{N}\frac{H_nC_n(x)\mathscr{F}_n}{n(1+n)} &\approx \cot\left(\frac{x}{2}\right)f(x). \end{aligned} \tag{39}$$

To find unknown quantities, $\mathscr{G}_m$ and $\mathscr{F}_m$, both sides of these equations are multiplied by $\cot(x/2)C_m(x)$ and then integrated to obtain the result

$$\begin{aligned} \mathscr{G}_m + \frac{2}{\pi}\sum_{n=1}^{N}\frac{I_{mn}H_n\mathscr{G}_n}{n(1+n)} &= G_m, \\ \mathscr{F}_m + \frac{2}{\pi}\sum_{n=1}^{N}\frac{I_{mn}H_n\mathscr{F}_n}{n(1+n)} &= F_m. \end{aligned} \tag{40}$$

These two sets of linear equations are then solved to find $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_N$ and $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_N$. Finally the constant η is found

$$\eta \approx -\left\{\frac{C_1(\alpha) - \frac{2}{\pi}\sum_{n=1}^{N}\frac{H_n\mathscr{G}_nC_n(\alpha)}{n(1+n)}}{f(\alpha) - \frac{2}{\pi}\sum_{n=1}^{N}\frac{H_n\mathscr{F}_nC_n(\alpha)}{n(1+n)}}\right\}, \tag{41}$$

where the condition $h_0(\alpha) + \eta h_1(\alpha) = 0$ has been used.

4. Axisymmetric flow

The coefficient B_n is associated with the axial flow and the boundary conditions extracted from Eqs. (9) and (10) are written as

$$\sum_{n=1}^{\infty} n(1+n)(1+H_n)B_nP_n(\cos\theta) + P_1(\cos\theta) = 0, \quad 0 < \theta < \alpha, \tag{42}$$

$$\sum_{n=0}^{\infty}(1+2n)B_nP_n(\cos\theta) + \xi P_1(\cos\theta) = 0, \quad \alpha < \theta < \pi. \tag{43}$$

The aim is to find the coefficients $B_0, B_1, B_2, \ldots$ that satisfy the dual series and the following identity:

$$\sin\theta P_n^1(\cos\theta) = n(1+n)\int_0^\theta P_n(\cos t)\sin t\,\mathrm{d}t$$

is used on Eq. (42). Then differentiating equation (43) with respect to θ eliminates B_0 and gives the dual series as

$$\sum_{n=1}^{\infty}(1+H_n)B_nP_n^1(\cos\theta) + \frac{P_1^1(\cos\theta)}{2} = 0, \quad 0 < \theta < \alpha, \tag{44}$$

$$\sum_{n=1}^{\infty}(1+2n)B_nP_n^1(\cos\theta) + \xi P_1^1(\cos\theta) = 0, \quad \alpha < \theta < \pi. \tag{45}$$

These two series are identical to Eqs. (14) and (16) given in Section 3 if the constant η is set equal to zero. Upon repeating all the steps in Section 3 between Eqs. (24)–(29) it follows that a function $j(x)$ can be defined in the region $0 < x < \alpha$ by the relation

$$\sum_{n=1}^{\infty} B_nC_n(x) + \frac{\xi C_1(x)}{3} = \omega\tan\left(\frac{x}{2}\right)\begin{cases} j(x), & 0 < x < \alpha, \\ 0, & \alpha < x < \pi. \end{cases} \tag{46}$$

Using the orthogonality condition (20), the coefficients are found as

$$B_n = -\frac{\xi\delta_{1n}}{3} + \frac{2\omega}{\pi n(1+n)}\int_0^\alpha j(t)\cot\left(\frac{t}{2}\right)C_n(t)\mathrm{d}t, \tag{47}$$

where ω is given in Eq. (15). The integral equation satisfied by $j(x)$ is given by

$$j(x) + \frac{2}{\pi}\int_0^\alpha j(t)K(x,t)\mathrm{d}t = C_1(x)\cot\left(\frac{x}{2}\right) \quad 0 < x < \alpha, \tag{48}$$

where the kernel $K(x,t)$ is given in Eq. (34). As discussed previously, when $\beta = 1$ the solution for $j(x)$ is $j(x) \approx C_1(x)\cot\left(\frac{x}{2}\right)$ and using this as the first iterated solution the coefficient B_n is approximately given by

$$B_n \approx -\frac{\xi\delta_{1n}}{3} + \frac{2\omega G_n}{\pi n(1+n)}, \tag{49}$$

when $\beta = 1$ this agrees with Collins' result.

In general it is not possible to find the exact solution and an approximate solution is obtained by truncating the kernel at N terms. The coefficient B_n is written as

$$B_n = -\frac{\xi\delta_{1n}}{3} + \frac{2\omega\mathscr{G}_n}{\pi n(1+n)}, \tag{50}$$

where the terms $\mathscr{G}_n$ are found by solving the set of linear equations given in (40)

$$\mathscr{G}_m + \frac{2}{\pi}\sum_{n=1}^{N}\frac{I_{mn}H_n\mathscr{G}_n}{n(1+n)} = G_m. \tag{51}$$

Once the values of $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_N$ are found, the coefficients B_1, B_2,... can be deduced and the coefficient B_0 can be obtained by evaluating Eq. (43) at $\theta = \pi$.

5. Effective thermal conductivity

Most of the schemes that are used to find the effective thermal conductivity in a composite material containing randomly dispersed particles rely upon the thermal field associated with an isolated inhomogeneity embedded in an infinite matrix [3,5,10–13]. Now considering the expression for the matrix temperature in Eq. (6), it can be shown that in a suitable representative test volume the higher order series terms $n > 1$ make no contribution to the average flux and hence do not enter the calculation for the effective conductivity. Thus, using only the first series term gives the form

$$u = u^{\infty} - (\varepsilon_x Ax + \varepsilon_z Bz)\left(\frac{a}{r}\right)^3, \tag{52}$$

where $A = (1 + H_1)A_1$ and $B = (1 + H_1)B_1$. To find A_1 and B_1 a system of linear equations $N \times N$ needs to be solved. The convergence is rapid and setting N equal to 2 or 3 gives fairly accurate results.

If the cracked caps are aligned then the overall effective conductivity is anisotropic and can be found using Maxwell's [3] treatment. He considers a spherical region of radius equal to c which encloses M particles and the number M is related to the volume fraction f by $Ma^3 = fc^3$. This spherical region is enclosed by an infinite matrix having conductivity k_m. Provided that the interaction between particles is neglected, the effect of the M particles can be summed, and at large distances, that is when $r \gg c$, the temperature is approximately

$$u = u^{\infty} - f(\varepsilon_x Ax + \varepsilon_z Bz)\left(\frac{c}{r}\right)^3. \tag{53}$$

On the other hand if the composite region inside the radius c containing the M particles is replaced with homogeneous material having an effective conductivity $k_x^e = k_y^e \neq k_z^e$ then the temperature is

$$u = u^{\infty} - (\varepsilon_x \chi_x x + \varepsilon_z \chi_z z)\left(\frac{c}{r}\right)^3, \tag{54}$$

where χ_x and χ_z are

$$\chi_x = \frac{k_x^e - k_m}{k_x^e + 2k_m}, \quad \chi_z = \frac{k_z^e - k_m}{k_z^e + 2k_m}. \tag{55}$$

If the two expressions for the temperature are to be equal then $\chi_x = fA$ and $\chi_z = fB$. Solving for the conductivity components k_x^e and k_z^e yields the expressions

$$k_x^e = k_m\left\{\frac{1+2fA}{1-fA}\right\}, \quad k_z^e = k_m\left\{\frac{1+2fB}{1-fB}\right\}. \tag{56}$$

If there is no debonding and the interface is perfect, then the effect of the particle is of course isotropic and $A = B = \gamma$ where

$$\gamma = \frac{1-\beta}{1+2\beta} \Longleftrightarrow \beta = \frac{1-\gamma}{1+2\gamma} \tag{57}$$

and Maxwell's estimate for a dispersion of spherical particles is obtained.

5.1. Fictitious inclusion

The debonded spherical particle can be replaced by a fictitious anisotropic inclusion having a thermally perfect interface. A similar situation arises in elastic problems, when particles are partially debonded [12]. Suppose that the fictitious particle has thermal conductivity $k_x = k_y \neq k_z$ and this conductivity can be found using Eq. (56) by setting $f = 1$ to obtain

$$\lambda_x = \frac{k_x}{k_p} = \beta\left\{\frac{1+2A}{1-A}\right\}, \quad \lambda_z = \frac{k_z}{k_p} = \beta\left\{\frac{1+2B}{1-B}\right\}. \tag{58}$$

When the particle is perfectly bonded, $\alpha = 0$, the coefficients $A = B = \gamma$ and $\lambda_z = \lambda_x = 1$. On the other hand, for a completely debonded particle, i.e. $\alpha = \pi/2$, the fictious particle has $\lambda_z = \lambda_x = 0$. Graphs of k_x/k_p and k_z/k_p versus α are shown in Figs. 1 and 2, respectively. The various curves correspond to the indicated values of β and as might be expected, when the particle conductivity is high, i.e. small values of β, the values of λ_x, λ_z decay rapidly. On the other hand, if the particle has very low conductivity the influence of the debonding is not so pronounced.

If the orientation of the crack caps is random then the composite can be viewed as a randomly oriented dispersion of anisotropic spherical fictitious particles. The overall

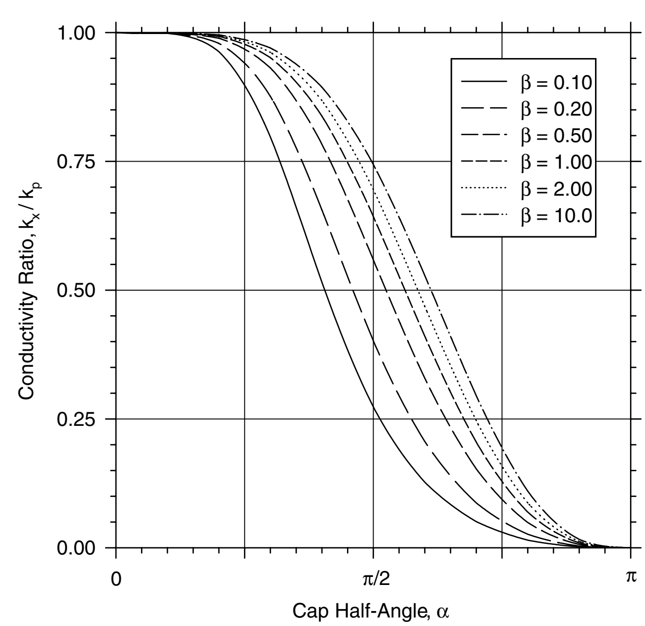

Fig. 1. The behaviour of the particle conductivity k_x/k_p as a function of the spherical-cap crack half-angle α. The various curves correspond to values of $\beta = k_m/k_p$ in the range $0.1 < \beta < 10$.

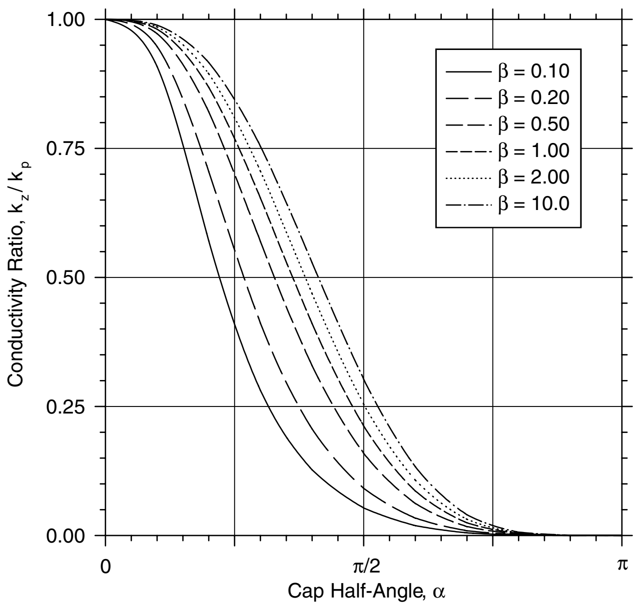

Fig. 2. The behaviour of the particle conductivity k_z/k_p as a function of the spherical-cap crack half-angle α. The various curves correspond to values of $\beta = k_m/k_p$ in the range $0.1 < \beta < 10$.

conductivity is isotropic and methods of treating this problem are described elsewhere [1,2,13].

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